Algebra I Back Paper

Answer all questions with carefully reasoned and written proofs. This exam is for 3 hours. Each question is worth 10 marks.

I. Let V and W be finite-dimensional vector spaces over F. Prove that there exists a onto linear transformation from V to W iff $dim(W) \leq dim(V)$.

II. Recall that an inner automorphism of G is one of the form $\tau(h) = ghg^{-1}$ where g is a fixed element of G. Show that the set of inner automorphisms of G is a group under composition and that this group is isomorphic to G/Z where Z is the centre of G.

III. Let G be a group of order 56. Prove that G has a non-trivial normal subgroup. You must fully state any result that you use.

IV. Let P be the vector space of polynomials of degree less than or equal to 3 in a single variable x. Let D be the linear transformation of "derivative with respect to x" from P to P. Find the matrix of D with respect to the basis $\{1, x+2, x^2+2x+3, x^3+2x^2+3x+4\}$. What are the change of bases matrices from this to the standard basis $\{1, x, x^2, x^3\}$ and the other way round ?

V. Let G be a group of order p^n where p is a prime and $n \geq 1$. Prove that the centre of G is not $\{1\}$.

VI. Prove that if V and W are finite-dimensional vector spaces over F and $T : V \to W$ is a linear transformation then there exist bases B of V and C of W with respect to which the matrix of T is of the form:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where the I is an identity matrix.

VII. Prove that if H and K are normal subgroups of G with $H \cap K = \{1\}$ and $HK = G$, then G is isomorphic to the product group $H \times K$.

VIII. Let $S = (v_1, v_2, \cdots, v_n)$ be an ordered set of vectors in a vector space V over the field F. Define the following:
(a) Span(S)
(b) Linear independence of S
(c) Basis
and prove that S is a basis of V iff each element of V has a unique expression of the form $c_1v_1 + \cdots c_nv_n$ where the $c_i \in F$.

IX. Prove that $Aut(\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^*$ after defining these objects.

X. Let U, V, W be finite dimensional vector spaces over F of dimensions m, n, p respectively and let A, B, C be bases respectively. Say $S : U \to V$ and $T : V \to W$ are linear transformations whose matrices with repect to the chosen bases are P, Q respectively. Prove that the matrix of $T \circ S$ with respect to the bases
A and C is given by QP.